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IN FUNCTION SPACE

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to Target Sets in Function Space

by

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Abstract

Optimal control of systems governed by functional differential equations of retarded and neutral type is considered. Problems with function space initial and terminal manifolds are investigated. Existence of optimal controls, regularity, and bang-bang properties are discussed. Necessary and sufficient conditions are derived and several solved examples which illustrate the theory are presented.

Section 1. Introduction.

A large number of papers have been written on the control of functional differential equations to target sets in R^n (for partial bibliographies see [4, 5, 44]). In this paper we treat a number of aspects concerning control to targets in function space. To the authors' knowledge only in the recent investigations reported in [36, 38, 39, 40, 49, 54] have others reported results for such problems. Popov [49] and Weiss [54] investigate controllability while the work of Jacobs and Kao [36, 38] concerns necessary and sufficient conditions for retarded systems. As is pointed out in Section 5 below, their methods are quite different from those employed here. We treat control problems involving a fairly general class of neutral functional differential systems, a class which includes as special cases almost all nonlinear retarded systems that are of interest. Control of neutral systems in a somewhat different formulation has been investigated by Kamenskii and Khvilon [37] who use the methods of Pontryagin, et. al. [48] to derive necessary conditions for problems with targets sets in R^n . In Section 5 we utilize the methods of Neustadt [46, 47] and Gamkrelidze [21] to obtain necessary conditions for the general nonlinear problem formulated in Section 2. We also show there that under certain convexity assumptions these conditions are sufficient for normal problems with linear-in-the-state systems.

In addition to the problem formulation, Section 2 also contains a motivating example which shows that boundary control problems for certain

hyperbolic systems can be transformed to problems involving control of neutral functional differential equations to function space targets. Section 3 contains some of the theory (existence, representation, etc.) of neutral systems which has been developed, for the formulation used in this paper, mainly by Hale and his students and colleagues [24, 25, 26, 30]. In Section 4 we present existence results for a large class of linear-in-the-state control problems. The related questions of smoothness (regularity) of controls and bang-bang properties (or lack thereof) are discussed. The paper is concluded with a section containing two solved examples along with comments concerning methods for solving examples via problem reformulation. More general necessary conditions plus a number of other solved examples may be found in the thesis of Kent [39, see also 40] on which much of the work reported here is based.

The following notational conventions will be adopted throughout the paper. We denote by $\mathcal{C}^n[a,b]$ the space $C([a,b], \mathbb{R}^n)$ of \mathbb{R}^n -valued continuous functions with the usual sup topology and by $L_p[a,b]$ the usual spaces of functions f (equivalence classes) with $|f|^p$ integrable in the sense of Lebesgue. We shall not use different symbols for various norms but let $|\cdot|$ represent the norm in whatever space may be appropriate. For example, if $x \in \mathcal{C}^n[a,b]$, $|x|$ is the sup norm of x while $|x(t)|$ represents the \mathbb{R}^n norm of $x(t)$. The symbol $\mathcal{L}_{n \times p}$ will denote the vector space of $n \times p$ real matrices and E_p will be used for the identity in the space $\mathcal{L}_{p \times p}$. $BV[a,b]$ will represent the space of functions of bounded

variation on $[a,b]$ with norm $|g| = \text{Var}([a,b];g) + |g(b)|$ where $\text{Var}([a,b];g)$ is the total variation of g on $[a,b]$.

If $x: [a-h,b] \rightarrow \mathbb{R}^n$, for $t \in [a,b]$ we denote by x_t the elements of $\mathcal{L}^n[-h,0]$ given by $x_t(\theta) = x(t+\theta)$, $\theta \in [-h,0]$. For systems involving hereditary dependence we shall use the notation $f(x(\cdot),t)$ to mean that $f: \mathcal{L}^n[t_0-h,t_1] \times [t_0,t_1] \rightarrow \mathbb{R}^n$ may depend on any or all of the values $x(s)$, $t_0-h \leq s \leq t$, where $t \in [t_0,t_1]$. Examples of such dependence are $f(x(\cdot),t) = G(x(t),x(t-h),t)$, $f(x(\cdot),t) = G(x_t,t)$, $f(x(\cdot),t) = \int_{t_0-h}^t a(t,s)G(x(s),t)ds$ (see [2, 24]).

Unless it is otherwise explicitly stated, all statements involving the concept of measure will be interpreted with respect to Lebesgue measure. All integrals will be Lebesgue or Lebesgue - Stieltjes integrals [15]. Finally we shall never distinguish between a vector and its transpose since in any vector-matrix operations it will be clear what is meant.

Section 2. Problem Formulation.

Let $\mathcal{T}_0, \mathcal{T}_1$ be given subsets of $\mathcal{L}^n[-h, 0]$ and suppose U is a specified non-empty subset of R^m . Define $\mathcal{U} = \{u: [t_0, t_1] \rightarrow R^m \mid u \text{ is bounded, measurable with } u(t) \in U \text{ for } t \in [t_0, t_1]\}$. We shall consider the general problem of minimizing $J = \int_{t_0}^{t_1} f^0(x(t), u(t), t) dt$ subject to

$$(2.1) \quad \frac{d}{dt} D(x(\cdot), t) = f(x(\cdot), u(t), t) \quad t \in [t_0, t_1]$$

$$x_{t_0} \in \mathcal{T}_0, \quad x_{t_1} \in \mathcal{T}_1$$

$$u \in \mathcal{U}$$

where the function D is defined by

$$(2.2) \quad D(x(\cdot), t) = x(t) - \int_{t_0-h}^t d_s \mu(t, s) x(s).$$

With $\mu \not\equiv 0$ and the hypotheses specified in Section 3 below, the system (2.1) is a functional differential equation (FDE) of neutral type. If $\mu \equiv 0$, the system is an FDE of retarded type. Simple examples of the type under consideration here are the differential difference equations

$$(2.3) \quad \dot{x}(t) - A(t)\dot{x}(t-h) = B(t)x(t) + C(t)x(t-h) + k(u(t), t)$$

and

$$(2.4) \quad \dot{x}(t) = B(t)x(t) + C(t)x(t-h) + k(u(t), t).$$

Many of the results obtained below can be extended to include certain types of systems involving a hereditary dependence on the control u in addition to the state x (see Kent [39]) but we shall not pursue that aspect of the problem in this paper.

There are a number of physical situations which motivate the problem as formulated above, although we shall cite only two of these here. Perhaps the simplest example where one desires to specify a terminal target in $\mathcal{C}^n[-h,0]$ involves systems such as (2.3), (2.4). It has been recognized for many years that the true "state" for such systems is x_t , not $x(t)$. If $x(t)$ represents some error which one wishes to be driven to zero (and held there if possible) and if the error is described by (2.3) or (2.4), then it is obvious that the desired terminal condition is $x_{t_1} = 0$.

A second motivational example which we shall only sketch here (see [39] for discussion of a similar example) involves boundary control of linear hyperbolic partial differential equations. Suppose we are given the wave equation for $w(t,x)$

$$(2.5) \quad w_{tt} - c^2 w_{xx} = 0 \quad t \in [0,T], x \in [0,1]$$

with boundary conditions

$$(2.6) \quad \begin{aligned} A_0(t)w_t(t,0) + B_0(t)w_x(t,0) &= g_0(t,w(t,0)) \\ A_1(t)w_t(t,1) + B_1(t)w_x(t,1) &= g_1(t,w(t,1)) \end{aligned}$$

and initial-terminal conditions

$$\begin{aligned}
 (2.7) \quad w(0, x) &= \alpha_0(x) & w_t(0, x) &= \alpha_1(x) \\
 w(T, x) &= \beta_0(x) & w_t(T, x) &= \beta_1(x).
 \end{aligned}$$

Suppose that A_i, B_i are continuously differentiable, g_i are absolutely continuous in t , continuously differentiable in w with g_{it} being dominated by L_2 functions, $i = 1, 2$. In addition assume that $\alpha'_0, \alpha_1, \beta'_0, \beta_1$ are absolutely continuous with L_2 derivatives ($' = \frac{d}{dx}$). Under the additional hypotheses $A_0(t) - \frac{1}{c} B_0(t) \neq 0$, $A_1(t) + \frac{1}{c} B_1(t) \neq 0$ for $t \in [0, T]$, one can derive an equivalent neutral system in the following way. Assume a solution of the form (D'Alembert)

$$w(t, x) = \varphi\left(t + \frac{x}{c}\right) + \psi\left(t - \frac{x}{c}\right).$$

Upon substitution in (2.6), followed by differentiation with respect to t and a few algebraic manipulations, one obtains a neutral system in $(\varphi', \psi') = (y, z)$ of the form

$$\begin{aligned}
 (2.8) \quad \dot{y}(t) + R(t)\dot{z}\left(t - \frac{2}{c}\right) &= H_1(t, y(\cdot), z(\cdot)) \\
 \dot{z}(t) + S(t)\dot{y}\left(t - \frac{2}{c}\right) &= H_2(t, y(\cdot), z(\cdot)).
 \end{aligned}$$

The data given in (2.7) can be used to produce initial and terminal data in terms of (y, z) for the system (2.8). Appropriate assumptions on the

boundary terms g_0, g_1 (which contain the controls for the problem) lead to a controlled system involving (2.8) for $t \in [\frac{1}{c}, T]$ with initial and terminal values of y specified on $[0, \frac{1}{c}]$ and at $t = T$ and corresponding values of z given on $[-\frac{1}{c}, \frac{1}{c}]$ and $[T - \frac{2}{c}, T]$. The terms H_1, H_2 are such that this initial data is sufficient to solve (2.8) for absolutely continuous (φ', ψ') having L_2 derivatives. It is not difficult to argue that this (φ, ψ) used in the D'Alembert solution above yields a solution to the original equation (2.5) in the (non-classical) sense that $w(t, x) = \varphi(t + \frac{x}{c}) + \psi(t - \frac{x}{c})$ is continuously differentiable with w_t, w_x being absolutely continuous and possessing L_2 partials satisfying (2.5) a.e.

The boundary conditions (2.6) include as special cases the usual boundary conditions [13, 14, 53] associated with (2.5) for transverse vibrations of a string or longitudinal vibrations in an elastic rod with elastically supported ends.

Other authors have pointed out connections between the study of hyperbolic systems and neutral FDE's. Brayton [9] and Slemrod [52] were concerned with systems arising from the study of lossless transmission lines while Cooke and Krumme [12] discussed a general method for reducing linear hyperbolic systems with nonlinear initial-boundary conditions to functional differential systems of neutral type.

Section 3. Representation Results for Linear Neutral Systems.

In this section we shall present properties of solutions of neutral systems which will be needed in the ensuing discussions. Our main results pertain to the variation of parameters representation for solutions to general linear systems. Referring to the function D defined in (2.2), we make the following standing assumptions on $\mu: \mathbb{R}^2 \rightarrow \mathcal{L}_{n \times n}$.

(3.1) $\mu(\sigma, \theta) = 0$ for $\theta \geq \sigma$, $\mu(\sigma, \theta) = \mu(\sigma, t_0 - h)$ for $\theta < t_0 - h$; μ is Borel measurable, continuous from the right in its first argument and continuous from the left in its second argument; $\theta \rightarrow \mu(\sigma, \theta)$ is of bounded variation on every finite θ interval, uniformly in σ ; and the mapping $t \rightarrow \Gamma(\varphi, t) \equiv \int_{t_0 - h}^t d_s \mu(t, s) \varphi(s)$ is continuous on $[t_0, t_1]$ for each fixed $\varphi \in \mathcal{C}^n[t_0 - h, t_1]$, which obviously implies that $(\varphi, t) \rightarrow \Gamma(\varphi, t)$ is continuous.

(3.2) there is a continuous non-decreasing function δ with $\delta(0) = 0$ such that for each $t \in \mathbb{R}^1$ and $\varepsilon > 0$ we have $\text{Var}([t - \varepsilon, t]; \mu(t, \cdot)) \leq \delta(\varepsilon)$.

Specific conditions on μ directly for which the last hypothesis in (3.1) obtains have been given by Kent [39]. Included as a special case of these is the situation where $\mu(s, \theta) = \mathcal{J}(s, \theta) + \mathcal{A}(s, \theta)$, $\theta \rightarrow \mathcal{J}(s, \theta)$ being a "well-behaved" jump function and $\theta \rightarrow \mathcal{A}(s, \theta)$ representing the absolutely continuous part of $\theta \rightarrow \mu(s, \theta)$. We shall not present the exact

technical assumptions on \mathcal{L}, \mathcal{A} here, but refer the interested reader to [39]. It suffices to remark that systems encountered in applications almost always satisfy these conditions.

We next consider solutions to

$$(3.3) \quad \begin{aligned} \frac{d}{dt} D(x(\cdot), t) &= \int_{t_0-h}^t d_s \eta(t, s) x(s) + g(t) & t \in [t_0, t_1] \\ x_{t_0} &= \varphi, \end{aligned}$$

where by a solution x we shall mean an $x \in \mathcal{C}^n[t_0-h, t_1]$ such that $t \rightarrow D(x(\cdot), t)$ is absolutely continuous on $[t_0, t_1]$ with (3.3) being satisfied a.e. The non-homogeneous term $g: [t_0, t_1] \rightarrow \mathbb{R}^n$ will always satisfy $g \in L_1$ and we make the following hypotheses on $\eta: \mathbb{R}^2 \rightarrow \mathcal{L}_{n \times n}$.

$$(3.4) \quad \begin{aligned} \eta(\sigma, \theta) &= 0 \text{ for } \theta \geq \sigma, \eta(\sigma, \theta) = \eta(\sigma, t_0-h) \text{ for } \theta < t_0-h; \eta \text{ is} \\ &\text{measurable, continuous from the left in its second variable on} \\ &(-\infty, \sigma); \theta \rightarrow \eta(\sigma, \theta) \text{ is of bounded variation on every finite } \theta \text{ inter-} \\ &\text{val and there is an } m \in L_1^{\text{loc}} \text{ such that } \text{Var}([t_0-h, \sigma]; \eta(\sigma, \cdot)) \leq m(\sigma). \end{aligned}$$

Under the above hypotheses and Caratheodory type assumptions on f , one can prove the usual local existence and continuation theorems for solutions to (2.1) with $x_{t_0} = \varphi, \varphi \in \mathcal{C}^n[-h, 0]$. In addition one can establish that (3.3) possesses a unique solution [25, 26, 30, 39]. We turn next to the "adjoint" system to (3.3) with $g \equiv 0$.

Theorem 3.1. Under the assumptions (3.1), (3.2), (3.4), for each fixed $t \in [t_0, t_1]$ the system

$$(3.5) \quad \begin{aligned} Y(s, t) &= E_n + \int_s^{t^+} d_\alpha Y(\alpha, t) \mu(\alpha, s) - \int_s^t Y(\alpha, t) \eta(\alpha, s) d\alpha \quad s \in [t_0, t) \\ Y(t, t) &= E_n, \quad Y(s, t) = 0 \quad \text{for } s > t \end{aligned}$$

has a unique solution on $[t_0, t_1]$. This solution $Y(s, t) \in \mathcal{L}_{n \times n}$ is left-continuous in its first argument and $|Y(s, t)| \leq \mathcal{B}$, $\text{Var}([t_0, t_1]; Y(\cdot, t)) \leq \mathcal{B}$ for $(s, t) \in [t_0, t_1] \times [t_0, t_1]$ where \mathcal{B} is finite and independent of (s, t) .

Proof: We assume for ease in notation (and without loss of generality) that $t_0 = 0$. The proof of existence of a unique solution and left continuity in its first argument is due to Henry [30]. We shall here only sketch the arguments, indicating how one obtains the bound \mathcal{B} . We note that it suffices to prove the uniform bound on the variation of $Y(\cdot, t)$ since for $s \in [t_0, t_1]$

$$\begin{aligned} \text{Var}([t_0, t_1]; Y(\cdot, t)) &\geq |Y(t_0, t) - Y(s, t)| + |Y(s, t) - Y(t_1, t)| \\ &\geq |Y(s, t)| - |Y(t_1, t)| \geq |Y(s, t)| - |E_n|. \end{aligned}$$

In the proof sketched here, one actually obtains existence of the solution to (3.5) on $|s| \leq t_1$, $|t| \leq t_1$. Let $\varepsilon > 0$ be chosen sufficiently small so that

$$(3.6) \quad \delta(\mathcal{E}) + \int_{t-\mathcal{E}}^t m(\theta) d\theta \leq \lambda < 1$$

for all $|t| \leq 2t_1$, where δ is the function guaranteed in (3.2). We make the induction hypotheses (clearly true for $p = 0$) that for $|t| \leq t_1$ the solution $Y(s, t)$ of (3.5) exists for $s \in [t - p\mathcal{E}, t_1]$ and satisfies $\text{Var}([t - p\mathcal{E}, t_1]; Y(\cdot, t)) \leq K_p$ where K_p is independent of t . We then define successive approximants by

$$Y^0(s, t) = \begin{cases} Y(s, t) & s \in (t - p\mathcal{E}, t_1] \\ Y(t - p\mathcal{E}, t) & s \in [-t_1, t - p\mathcal{E}] \end{cases}$$

and

$$Y^k(s, t) = \begin{cases} Y(s, t) & s \in (t - p\mathcal{E}, t_1] \\ E_n + \int_s^{t+} d_\alpha Y^{k-1}(\alpha, t) \mu(\alpha, s) - \int_s^t Y^{k-1}(\alpha, t) \eta(\alpha, s) d\alpha & s \in [-t_1, t - p\mathcal{E}] \end{cases}$$

for $k = 1, 2, \dots$. Using (3.6) and these definitions along with the hypotheses in (3.2), (3.4), one can easily show that

$$\|Y^{k+1}(\cdot, t) - Y^k(\cdot, t)\| \leq \lambda \|Y^k(\cdot, t) - Y^{k-1}(\cdot, t)\|$$

and hence

$$\|Y^{k+1}(\cdot, t) - Y^k(\cdot, t)\| \leq \lambda^k \|Y^1(\cdot, t) - Y^0(\cdot, t)\|$$

where $\|g\| \equiv \text{Var}([t-(p+1)\mathcal{E}, t-p\mathcal{E}]; g)$. Observing that one actually has $Y^k(s, t) = Y(s, t)$ for $s \in [t-p\mathcal{E}, t_1]$, one sees that $Y^k(\cdot, t)$ converges in $BV[t-(p+1)\mathcal{E}, t-p\mathcal{E}]$ to a function $\tilde{Y}(\cdot, t)$. Letting $k \rightarrow \infty$ in the above approximants we see that this extends the solution from $[t-p\mathcal{E}, t_1]$ to $[t-(p+1)\mathcal{E}, t_1]$. A finite number of induction steps on p yields existence as claimed. The left continuity and uniqueness follow directly from the hypotheses on μ, η and the equation for Y .

Returning to the arguments involving $Y^k(\cdot, t)$, we have that

$$\begin{aligned} \|Y^k(\cdot, t)\| &\leq \|Y^k(\cdot, t) - Y^{k-1}(\cdot, t)\| + \|Y^{k-1}(\cdot, t)\| \\ &\leq \{\lambda^{k-1} + \lambda^{k-2} + \dots + \lambda^1 + 1\} \|Y^1(\cdot, t) - Y^0(\cdot, t)\| + \|Y^0(\cdot, t)\| \\ &\leq \frac{1}{1-\lambda} \|Y^1(\cdot, t) - Y^0(\cdot, t)\| + \|Y^0(\cdot, t)\|. \end{aligned}$$

But $\|Y^0(\cdot, t)\| = 0$ and using elementary arguments with the hypotheses on η, μ , the definitions of Y^0, Y^1 , one can easily show that

$$\|Y^1(\cdot, t) - Y^0(\cdot, t)\| \leq K_p \{M + 2 \int_{t-(p+1)\mathcal{E}}^{t-p\mathcal{E}} m(\theta) d\theta\}$$

for $|t| \leq t_1$. It follows that

$$\|Y^k(\cdot, t)\| \leq B_p$$

where B_p is independent of t , $|t| \leq t_1$. Since $\|Y^k(\cdot, t) - \tilde{Y}(\cdot, t)\| \rightarrow 0$, we obtain $\|\tilde{Y}(\cdot, t)\| \leq B_p$. Thus

$$\begin{aligned} \text{Var}([t-(p+1)\varepsilon, t_1]; Y(\cdot, t)) &\leq \|\tilde{Y}(\cdot, t)\| + \text{Var}([t-p\varepsilon, t_1]; Y(\cdot, t)) \\ &\leq B_p + K_p = K_{p+1} \end{aligned}$$

where K_{p+1} is independent of t , $|t| \leq t_1$. The finite number of induction steps on p then produce the bound independent of t .

Theorem 3.2. Let x be the solution of (3.3) under the assumptions (3.1), (3.2), (3.4). Then for $t \in [t_0, t_1]$

$$\begin{aligned} (3.7) \quad x(t) &= Y(t_0, t)D(\varphi, t_0) + \int_{t_0-h}^{t_0} d_\beta \gamma(t, \beta)\varphi(\beta) \\ &\quad + \int_{t_0}^t Y(\beta, t)g(\beta)d\beta \end{aligned}$$

where Y is given by (3.5) and

$$\gamma(t, \beta) \equiv -\int_{t_0}^{t^+} d_\alpha Y(\alpha, t)\mu(\alpha, \beta) + \int_{t_0}^t Y(\alpha, t)\eta(\alpha, \beta)d\alpha.$$

The proof of this theorem is due to Henry [30]. We shall omit it here since it involves a standard type of argument making use of integration by parts, an unsymmetric Fubini theorem [10], and the equation for Y . We note that for $\mu \equiv 0$ the adjoint system (3.5) and the representation (3.7) reduce to that for retarded systems [2, 3].

Remark 3.1. We make some further comments about the solution Y of (3.5) which may correctly be regarded as a "fundamental matrix solution". It is not difficult to show [39] that for fixed s , the function $t \rightarrow Y(s, t)$ (which is, in general, discontinuous) is BV and satisfies

$$Y(s, t) = E_n + \int_s^t d_\theta \mu(t, \theta) Y(s, \theta) + \int_s^t d\beta \int_s^\beta d_\theta \eta(\beta, \theta) Y(s, \theta)$$

for $t \geq s$, with $Y(s, t) = 0$ for $t < s$. Note that this is just the integrated form of (3.3) with $g \equiv 0$. In a subsequent section (Section 5) of this paper, it will be essential that the mapping $(s, t) \rightarrow Y(s, t)$ be Borel measurable. (This is needed in order to use Y as the measure in the unsymmetric Fubini theorem [10].) This will be true under varied assumptions on μ . For example, Kent [39] has shown that it is sufficient that $t \rightarrow \mu(t, \theta)$ be of bounded variation on each finite t interval for each fixed θ . Henry [30] has established Borel measurability of Y under other assumptions. Since we do not wish to become involved in these technical details here and since any system of interest to us in this paper would meet either Kent's or Henry's assumptions, we make the standing hypothesis that $(s, t) \rightarrow Y(s, t)$ is Borel measurable.

Before presenting the final results of this section we must make the following definition. For $\Phi \subset \mathcal{C}^n[-h, 0]$, $K \in L_1[t_0, t_1]$ and $X \subset \mathbb{R}^n$, define $C([t_0 - h, t_1], X; \Phi, K) \subset \mathcal{C}^n[t_0 - h, t_1]$ by $C([t_0 - h, t_1], X; \Phi, K) = \{x \in \mathcal{C}^n[t_0 - h, t_1] \mid x_{t_0} \in \Phi, x(t) \in X \text{ for } t \in [t_0 - h, t_1], \left| \frac{d}{dt} D(x(\cdot), t) \right| \leq K(t) \text{ a.e. } t \in [t_0, t_1]\}$.

Theorem 3.3. In addition to (3.1), (3.2), assume that μ satisfies

there exist $\ell > 0$, $L > 0$ such that for $s \leq t$

$$(3.8) \quad \int_{t-\ell}^t |d_{\theta}\{\mu(t, \theta) - \mu(s, \theta)\}| \leq L|t-s|.$$

Then X, Φ compact implies $C([t_0-h, t_1], X; \Phi, K)$ is a compact subset of $\mathcal{C}^n[t_0-h, t_1]$. It is also convex if X, Φ are.

Proof: Convexity follows from the linearity of $D(x(\cdot), t) = x(t) - \Gamma(x, t)$.

For any $\varphi \in \Phi$, we define x_{φ} by $x_{\varphi}(t) = \varphi(t-t_0)$, $t \in [t_0-h, t_0]$, $x_{\varphi}(t) = \varphi(0)$, $t > t_0$. Then for $x \in C([t_0-h, t_1], X; \Phi, K)$ with $x_{t_0} = \varphi$ we have

$$\begin{aligned} |x(t) - x(t_0)| &\leq |D(x(\cdot), t) - D(x(\cdot), t_0)| + |\Gamma(x, t) - \Gamma(x_{\varphi}, t)| \\ &\quad + |\Gamma(x_{\varphi}, t) - \Gamma(x_{\varphi}, t_0)| \\ &\leq \int_{t_0}^t K(s) ds + \left| \int_{t_0}^t d_{\theta}\mu(t, \theta) \{x(\theta) - x(t_0)\} \right| \\ &\quad + |\Gamma(x_{\varphi}, t) - \Gamma(x_{\varphi}, t_0)|. \end{aligned}$$

Choosing p such that $0 < p \leq \ell$ and $\delta(p) < 1$, we thus find for $t \in [t_0, t_0+p]$

$$\begin{aligned} |x(t) - x(t_0)| &\leq \int_{t_0}^t K(s) ds + \delta(p) \|x - x_{\varphi}\|_t \\ &\quad + \sup\{|\Gamma(x_{\psi}, t) - \Gamma(x_{\psi}, t_0)| : t \in [t_0, t_0+p], \psi \in \Phi\} \end{aligned}$$

where $\|x\|_t = \sup\{|x(s)| : s \in [t_0-h, t]\}$. But since the right side of this expression is non-decreasing in t , we obtain

$$\|x-x_\varphi\|_t \leq \int_{t_0}^t K(s)ds + \delta(p)\|x-x_\varphi\|_t + \sup_{t,\psi} |\Gamma(x_\psi, t) - \Gamma(x_\psi, t_0)|$$

or

$$\|x-x_\varphi\|_t \leq b \left\{ \int_{t_0}^{t_0+p} K(s)ds + \sup_{t,\psi} |\Gamma(x_\psi, t) - \Gamma(x_\psi, t_0)| \right\} = R,$$

where $b = 1/(1-\delta(p))$.

We remark that the sup term is finite since Γ is continuous, Φ is compact.

For $t_0 \leq \tau \leq t \leq t_0+p$ we therefore have

$$\begin{aligned} |x(t) - x(\tau)| &\leq |D(x(\cdot), t) - D(x(\cdot), \tau)| + |\Gamma(x, t) - \Gamma(x, \tau)| \\ &\leq \int_{\tau}^t K(s)ds + |\Gamma(x_\varphi, t) - \Gamma(x_\varphi, \tau)| \\ &\quad + \left| \int_{t_0}^t d_s [\mu(t, s) - \mu(\tau, s)] \{x(s) - x_\varphi(s)\} \right| \\ &\leq \int_{\tau}^t K(s)ds + \sup\{|\Gamma(x_\psi, t) - \Gamma(x_\psi, \tau)| : \psi \in \Phi\} \\ &\quad + L|t - \tau|R. \end{aligned}$$

From the continuity of Γ and the compactness of Φ , it follows that the elements of $C([t_0-h, t_1], X; \Phi, K)$ form an equicontinuous family on $[t_0-h, t_0+p]$.

Since the restrictions of these elements to $[t_0-h, t_0+p]$ constitute a bounded subset of $\mathcal{S}^n[t_0-h, t_0+p]$, use of the Arzela-Ascoli theorem here, followed by repetition of the above arguments on $[t_0+p, t_0+2p]$, $[t_0+2p, t_0+3p]$, ... (for a finite number of steps), leads to the conclusion that $C([t_0-h, t_1], X; \Phi, K)$ is a conditionally compact subset of $\mathcal{S}^n[t_0-h, t_1]$. Using the compactness of X, Φ and the continuity of D , it is not difficult to argue that $C([t_0-h, t_1], X; \Phi, K)$ is a closed, hence compact, subset of $\mathcal{S}^n[t_0-h, t_1]$.

Section 4. Existence, Regularity and Bang-Bang Results.

We shall consider first systems that are linear in the state, i.e., system (3.3) with $g(t) = k(u(t), t)$. We shall approach the questions of existence, smoothness of controls and bang-bang properties by considering attainable sets in $\mathcal{L}^n[-h, 0]$. We assume

(4.1) the mapping $k: R^m \times R^1 \rightarrow R^n$ is continuous, the set $U \subset R^m$ is compact, and $k(U, t) \equiv \{k(u, t) \mid u \in U\}$ is convex for each t .

Define the family \mathcal{F} by $\mathcal{F} = \{f: \mathcal{L}^n[t_0-h, t_1] \times [t_0, t_1] \rightarrow R^n \mid f(x(\cdot), t) = \int_{t_0-h}^t d_s \eta(t, s)x(s) + k(u(t), t), u \in \mathcal{U}\}$ where \mathcal{U} is the class of admissible controls as defined in Section 2. We are thus considering

$$(4.2) \quad \begin{aligned} \frac{d}{dt} D(x(\cdot), t) &= f(x(\cdot), t) \quad t \in [t_0, t_1] \\ x_{t_0} &= \varphi \end{aligned}$$

for $(\varphi, f) \in \Phi \times \mathcal{F}$. We assume of course throughout that μ, η satisfy conditions (3.1), (3.2), (3.4). Recalling that $\|y\|_t = \sup\{|y(s)| : s \in [t_0-h, t]\}$ we then have

Lemma 4.1. In addition to the above hypotheses, assume Φ is a bounded subset of $\mathcal{L}^n[-h, 0]$. Then there is an $M > 0$ such that $\|x(\varphi, f)\|_t \leq M$ for $t \in [t_0, t_1]$, $(\varphi, f) \in \Phi \times \mathcal{F}$, where $x(\varphi, f)$ denotes the solution to (4.2) for $(\varphi, f) \in \Phi \times \mathcal{F}$.

Proof: It is easy to see that there is a constant \tilde{d} and an L_1 function \tilde{m} such that $|f(x(\cdot), t)| \leq \tilde{m}(t)[\|x\|_t + \tilde{d}]$ for every $f \in \mathcal{F}$. Letting $x = x(\varphi, f)$, $(\varphi, f) \in \Phi \times \mathcal{F}$, we have

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t d_s \mu(t, s) x(s) \\ &\quad + \int_{t_0-h}^t d_s [\mu(t, s) - \mu(t_0, s)] x(s) + \int_{t_0}^t f(x(\cdot), s) ds. \end{aligned}$$

Choosing p such $0 < \delta(p) < 1$ and letting $b = 1/(1 - \delta(p))$, we find for $t \in [t_0, t_0 + p]$

$$\|x\|_t \leq |\varphi| + \delta(p)\|x\|_t + 2\delta(t_1 - t_0 + h)|\varphi| + \int_{t_0}^t \tilde{m}(s)[\|x\|_s + \tilde{d}] ds.$$

It follows that

$$\tilde{d} + \|x\|_t \leq b \left\{ [1 + 2\delta(t_1 - t_0 + h)][|\varphi| + \tilde{d}] + \int_{t_0}^t \tilde{m}(s)[\|x\|_s + \tilde{d}] ds \right\}$$

and applying Gronwall's inequality, we find that for $t \in [t_0, t_0 + p]$

$$\tilde{d} + \|x\|_t \leq b\{1 + 2\delta(t_1 - t_0 + h)\}\{\tilde{d} + |\varphi|\} \exp\left[b \int_{t_0}^t \tilde{m}(s) ds\right].$$

Repetition of the above arguments on $[t_0 + p, t_0 + 2p], \dots, [t_0 + (N-1)p, t_1]$ yields

$$\tilde{d} + \|x\|_{t_1} \leq b^N \{1 + 2\delta(t_1 - t_0 + h)\}^N \{\tilde{d} + |\varphi|\} \exp\left[b \int_{t_0}^{t_1} \tilde{m}(s) ds\right].$$

The result then follows from the fact that Φ is bounded.

Lemma 4.2. Under the assumptions (3.1), (3.2), (3.4), (3.8) and (4.1), Φ compact in $\mathcal{C}^n[-h, 0]$ implies $\mathcal{A} \equiv \{x(\varphi, f) \mid (\varphi, f) \in \Phi \times \mathcal{F}\}$ is an equicontinuous subset of $\mathcal{C}^n[t_0-h, t_1]$.

Proof: From Lemma 4.1 it follows that for any $x \in \mathcal{A}$, $\frac{d}{dt} D(x(\cdot), t) \leq \tilde{m}(t)[\|x\|_t + \tilde{d}] \leq \tilde{m}(t)[M + \tilde{d}] = K(t)$ on $[t_0, t_1]$ where $K \in L_1$. The same lemma guarantees existence of a compact $X \subset \mathbb{R}^n$ such that $x(t) \in X$, $t \in [t_0-h, t_1]$, $x \in \mathcal{A}$. Thus \mathcal{A} is a subset of $C([t_0-h, t_1], X; \Phi, K)$ which, by Lemma 3.3, is a compact subset of $\mathcal{C}^n[t_0-h, t_1]$. The equicontinuity of \mathcal{A} thus follows from a well-known theorem [15, p. 266].

We define, for each $t \in [t_0, t_1]$, the attainable set at time t given by

$$\mathcal{A}_t(\Phi, \mathcal{F}) = \{z \in \mathcal{C}^n[-h, 0] \mid z = x_t(\varphi, f), (\varphi, f) \in \Phi \times \mathcal{F}\}.$$

Using the representation results given in Theorem 3.2 for the solutions $x(\varphi, f)$, we can write

$$\mathcal{A}_t(\Phi, \mathcal{F}) = \mathcal{R}_t(\Phi, 0) + \mathcal{R}_t(0, \mathcal{U})$$

where

$$\mathcal{R}_t(\Phi, 0) \equiv \left\{ y_t \in \mathcal{L}^n[-h, 0] \mid y(s) = Y(t_0, s)D(\varphi, t_0) + \int_{t_0-h}^{t_0} d_\beta Y(s, \beta) \varphi(\beta) \text{ for } s > t_0, y(s) = \varphi(s-t_0) \right. \\ \left. \text{for } s \in [t_0-h, t_0], \varphi \in \Phi \right\}$$

and

$$\mathcal{R}_t(0, \mathcal{U}) \equiv \left\{ y_t \in \mathcal{L}^n[-h, 0] \mid y(s) = \int_{t_0}^s Y(\beta, s) k(u(\beta), \beta) d\beta \text{ for } s > t_0, y(s) = 0 \text{ for } s \in [t_0-h, t_0], u \in \mathcal{U} \right\}.$$

We note that $\mathcal{R}_t(\Phi, 0)$ consists of restrictions of solutions to (3.3) with $g \equiv 0$ and initial data $\varphi \in \Phi$ while $\mathcal{R}_t(0, \mathcal{U})$ is the set of restrictions of solutions to (3.3) with $g(t) = k(u(t), t)$, $u \in \mathcal{U}$, and initial data $\varphi \equiv 0$.

For fixed $t \in (t_0, t_1]$, let us first consider the set $\mathcal{R}_t(0, \mathcal{U})$. We define the set $\mathcal{K}(\mathcal{U})$ and the mapping $T^t: L_1[t_0, t_1] \rightarrow \mathcal{L}^n[-h, 0]$ by

$$\mathcal{K}(\mathcal{U}) = \{g \in L_1[t_0, t_1] \mid g(s) = k(u(s), s), u \in \mathcal{U}\}$$

and

$$T^t(g)(\theta) = \begin{cases} \int_{t_0}^{t+\theta} Y(\beta, t+\theta) g(\beta) d\beta & t+\theta > t_0 \\ 0 & t+\theta \leq t_0 \end{cases}$$

for $\theta \in [-h, 0]$, so that $\mathcal{R}_t(0, \mathcal{U}) = T^t(\mathcal{K}(\mathcal{U}))$.

Lemma 4.3. Under the assumptions (4.1), $\mathcal{R}_t(0, \mathcal{U})$ is a closed subset of $\mathcal{L}^n[-h, 0]$.

Proof: The ideas in this proof are by now quite familiar to control theorists [32, p. 18-23; 11, 17, 31]. First, from arguments similar to Filippov's [19], it follows easily that $\mathcal{K}(\mathcal{U})$ is a weakly sequentially closed subset of $L_1[t_0, t_1]$. From the hypotheses (4.1) one obtains [15, p. 292] that $\mathcal{K}(\mathcal{U})$ is weakly sequentially compact in $L_1[t_0, t_1]$ and hence by Eberlein-Smulian [15, p. 430] the weak closure of $\mathcal{K}(\mathcal{U})$ is weakly compact. But the weak closure of $\mathcal{K}(\mathcal{U})$ is the same as its weak sequential closure [15, p. 434]. Hence $\mathcal{K}(\mathcal{U})$ is a weakly compact subset of $L_1[t_0, t_1]$.

The map T^t is clearly continuous with respect to the strong topologies of L_1 and $\mathcal{L}^n[-h, 0]$, and hence continuous with respect to the weak topologies on these spaces [15, p. 422]. It follows that $T^t(\mathcal{K}(\mathcal{U}))$ is weakly compact and hence weakly [a fortiori strongly] closed in $\mathcal{L}^n[-h, 0]$.

Remark 4.1. The conditions (4.1) under which Lemma 4.3 obtains can be relaxed. Using arguments similar to those of Jacobs [35, p. 416] one can show that $\mathcal{K}(\mathcal{U})$ is weakly sequentially closed under the assumptions: $t \rightarrow U(t)$ defines an upper semicontinuous mapping with range in the collection of non-empty compact subsets of R^m ; $k(U(t), t)$ is convex for each $t \in [t_0, t_1]$; and $u \rightarrow k(u, t)$ is continuous for each t , $t \rightarrow k(u, t)$ is measurable and there is an $m \in L_1$ such that $|k(u, t)| \leq m(t)$, $u \in U(t)$.

The other arguments in the proof of Lemma 4.3 then hold without change. Comments on relaxing the convexity assumptions will be made below.

Under the assumptions of Lemma 4.2, we see that $\mathcal{R}_t(0, \mathcal{U})$ is an equicontinuous subset of $\mathcal{C}^n[-h, 0]$. Since it is also bounded, the Arzela-Ascoli theorem implies that $\mathcal{R}_t(0, \mathcal{U})$ is conditionally compact. Lemma 4.3 then yields the compactness of $\mathcal{R}_t(0, \mathcal{U})$.

Next let us consider $\mathcal{R}_t(\Phi, 0)$. For Φ compact in $\mathcal{C}^n[-h, 0]$, to show $\mathcal{R}_t(\Phi, 0)$ compact in $\mathcal{C}^n[-h, 0]$ it suffices to show that the mapping $\varphi \rightarrow y_t(\varphi)$ is continuous, where y_t is as given in the definition of $\mathcal{R}_t(\Phi, 0)$. Clearly, it is enough to demonstrate that for $t \geq t_0 + h$ the mapping $\varphi \rightarrow z_t(\varphi) \in \mathcal{C}^n[-h, 0]$ is continuous, where

$$z_t(\varphi)(\theta) \equiv Y(t_0, t+\theta)D(\varphi, t_0) + \int_{t_0-h}^{t_0} d_{\beta} \gamma(t+\theta, \beta) \varphi(\beta) \quad \theta \in [-h, 0].$$

We have

$$\begin{aligned} |z_t(\varphi) - z_t(\psi)| &\leq \sup_{\theta \in [-h, 0]} |Y(t_0, t+\theta)| |D(\varphi, t_0) - D(\psi, t_0)| \\ &\quad + \sup_{\theta \in [-h, 0]} \left| \int_{t_0-h}^{t_0} d_{\beta} \gamma(t+\theta, \beta) [\varphi(\beta) - \psi(\beta)] \right| \\ &\leq \sup_{\theta} |Y(t_0, t+\theta)| |D(\varphi, t_0) - D(\psi, t_0)| \\ &\quad + \sup_{\theta} \text{Var}([t_0-h, t_0]; \gamma(t+\theta, \cdot)) |\varphi - \psi|. \end{aligned}$$

From the continuity of D , it suffices to show that the sup terms are finite. Using the definition of γ and the hypotheses (3.1), (3.4) on μ, η , one can

show easily that for $\theta \in [-h, 0]$

$$\begin{aligned} \text{Var}([t_0-h, t_0]; \gamma(t+\theta, \cdot)) &\leq M_1 \int_{t_0}^{t+\theta^+} |d_\alpha Y(\alpha, t+\theta)| \\ &\quad + \int_{t_0}^{t+\theta} |Y(\alpha, t+\theta)| m(\alpha) d\alpha. \end{aligned}$$

The bounds guaranteed in Theorem 3.1 then imply that the sup terms are indeed finite.

Theorem 4.1. Under the assumptions (3.1), (3.2), (3.4), (3.8), and (4.1), Φ compact in $\mathcal{C}^n[-h, 0]$ implies $\mathcal{A}_t(\Phi, \mathcal{F})$ is compact in $\mathcal{C}^n[-h, 0]$, $t \in [t_0, t_1]$. Furthermore, the mapping $t \rightarrow \mathcal{A}_t(\Phi, \mathcal{F})$ is continuous with respect to the Hausdorff metric [7].

Proof. The first conclusion of the theorem is evident from the compactness of $\mathcal{R}_t(\Phi, 0)$ and $\mathcal{R}_t(0, \mathcal{U})$. The second assertion follows easily from the fact (Lemma 4.2) that $t \rightarrow x_t(\varphi, f)$ is continuous uniformly in $(\varphi, f) \in \Phi \times \mathcal{F}$.

Remark 4.2. The continuity of $t \rightarrow \mathcal{A}_t(\Phi, \mathcal{F})$ can be proved for retarded systems ($\mu \equiv 0$) by the usual arguments ([42; p. 70-71], [45; p. 114]) involving the variation of parameters representation (Theorem 3.2). These arguments depend very much on the continuity of $t \rightarrow Y(\alpha, t)$. For neutral systems, this continuity requirement is not met and hence a direct extension of the usual arguments is not possible. The compactness arguments can also be made somewhat more directly for the retarded case by taking advantage of the

continuity of $t \rightarrow Y(\alpha, t)$.

The results given in Theorem 4.1 are sufficient to obtain existence theorems for a wide class of linear-in-the-state problems with initial and terminal manifolds in $\mathcal{E}^n[-h, 0]$. Since the proofs involve well-known arguments [32, 42, 45], we shall only list a few of these problems here. (In each case it is assumed that $\mathcal{T}_0 = \Phi$ is compact and \mathcal{T}_1 is closed.)

- (i) The time-optimal problem for hitting a target set $\mathcal{T}_1 \subset \mathcal{E}^n[-h, 0]$, starting from an initial manifold $\mathcal{T}_0 = \Phi$. (In the formulation of Section 2, take $f^0 = 1$ and t_1 not fixed.)
- (ii) The problem of minimizing $P(x_{t_1})$ subject to $x_{t_i} \in \mathcal{T}_i$, $i = 0, 1$, where $P: \mathcal{E}^n[-h, 0] \rightarrow \mathbb{R}^1$ is continuous.
- (iii) Minimization of $J = \int_{t_0}^{t_1} \{A^0 x(t) + k^0(u(t), t)\} dt$ subject to $x_{t_i} \in \mathcal{T}_i$, $i = 0, 1$. (The usual device of augmenting the system and then minimizing $P(\hat{x}) = x^0(t_1)$ allows a direct application of Theorem 4.1 here.)

Various generalizations are possible, e.g. allowing the target \mathcal{T}_1 to depend on time [4, 32, 42, 45]. We shall not pursue these matters in this paper since our main interest concerns problems with $\mathcal{T}_0, \mathcal{T}_1$ each consisting of a single point in $\mathcal{E}^n[-h, 0]$.

Kent has shown that fairly general existence results such as those by Jacobs [35] can be extended to include systems described by certain nonlinear neutral equations. Some of the arguments are tedious and we shall not present them here, but instead refer the reader to [39]. One type of nonlinear

existence result which we should mention here is needed for a complete discussion of Example 6.2 below. Briefly, suppose that in the formulation in

Section 2, f of (2.1) has the form $f(x(\cdot), u(t), t) = \int_{t_0-h}^t d_s \eta(t, s) x(s)$

+ $B(t)u(t)$, $f^0(x(t), u(t), t) = x(t)Qx(t) + u(t)Ru(t)$ where $Q \geq 0$, $R > 0$

and $\mathcal{T}_0 = \{\varphi\}$, $\mathcal{T}_1 = \{\zeta\}$ where φ, ζ are given in $\mathcal{C}^n[-h, 0]$. Existence results for this problem follow from standard arguments [42] which we shall

only sketch here. Letting $\{u_n\}$ be chosen, $u_n \in \mathcal{U}$, $x_{n_{t_0}} = \varphi$, $x_{n_{t_1}} = \zeta$, so

that $J(u_n) = \int_{t_0}^{t_1} x_n^T Q x_n + u_n^T R u_n \rightarrow \beta = \inf\{J(u) \mid u \in \mathcal{U}, x_{t_0}(u) = \varphi, x_{t_1}(u) = \zeta\}$,

one sees that $\int_{t_0}^{t_1} u_n^T R u_n$ is bounded, and hence $\{u_n\}$ is bounded in $L_2[t_0, t_1]$.

Choosing a weakly convergent subsequence $\{u_{n_k}\}$, it is not hard to argue that

the corresponding trajectories $\{x_{n_k}\}$ satisfy $x_{n_k}(t) \rightarrow x^*(t)$, $t \in [t_0, t_1]$,

where x^* is the trajectory corresponding to u^* , the weak limit of $\{u_{n_k}\}$.

Use of the weak lower-semi-continuity of $u \rightarrow \int u^T R u$ along with other well-

known arguments yields $J(u^*) \leq \beta$. If $U = R^m$, then $u^* \in \mathcal{U}^{++}$ and the proof

is completed. If U is compact convex in R^m , then [15, p. 422] some se-

quence of convex combinations of the u_{n_k} converges in $L_2[t_0, t_1]$ to u^* .

It follows that $u^*(t) \in U$ a.e. and again the proof is concluded.

We turn next to the questions of regularity (smoothness) of controls and bang-bang properties (U is taken compact in R^m). Many authors

⁺⁺In this case we take \mathcal{U} as $L_2[t_0, t_1]$ and do not insist that u^* be

pointwise bounded as required in the formulation of Section 2.

have investigated these questions for finite and certain types of infinite dimensional systems. An often considered question [18, 27, 28, 29, 32, 42] is the following: Given a "state" that is attainable from a given initial "state" employing a measurable control, is it possible to attain the same "state" using a piecewise continuous bang-bang control? An affirmative answer to this question has been given by Banks and Jacobs [4] for some classes of retarded systems when the attainable "states" are taken in R^n . The methods used are extensions of ideas due to Halkin and Hendricks [27, 28, 29]. The results in [4] can be extended to include certain types of neutral systems (for example, (2.3) above when A, B, C are analytic). However, our interest here is in attainable "states" in $\mathcal{L}^n[-h, 0]$. Since it is known that there is no infinite dimensional analogue of the Liapunov theorem which is the basis of the above arguments [27, 28, 29], we face considerable difficulties. In fact, in light of known results for linear ODE systems [see 32, p. 111] concerning bang-bang results for trajectories (attainable "states" in $\mathcal{L}^n[-h, 0]$ correspond to terminal segments of trajectories: $x(s), s \in [t_1 - h, t_1]$) one might suspect that the best that can be obtained is a density theorem. But what if one is willing to make assumptions on the FDE (such as A or C non-singular in (2.3) or (2.4)) so that ODE's are not special cases of the FDE? The authors did this for retarded systems, using the methods of Hale [23] involving a finite dimensional projection argument (for which the Liapunov type results are valid). Certain results on eigenfunction expansions of solutions to FDE's [6] were also needed since a limiting process in a space

of closed subsets of $\mathcal{S}^n[-h,0]$ was employed. This lead only to the density theorem one would certainly expect to be true. The following examples demonstrate the futility of our efforts.

Example 4.1. Consider the retarded system

$$\dot{x}(t) = x(t-1) + u(t) \quad t \in [0,3]$$

with $U = \{v \in \mathbb{R}^1 : |v| \leq 1\}$ and $\xi(t) = \frac{1}{2}(3-t)$, $t \in [2,3]$. It is not hard to show that ξ can be attained by a measurable control u with $u(t) \in U$, starting from the initial function $\phi(t) = -t$, $t \in [-1,0]$. But ξ is not attainable with a bang-bang control from any initial ϕ . Suppose it were, with the bang-bang control denoted by u^0 . Since $\dot{x}(t) = -\frac{1}{2}$ for $t \in [2,3]$ and $|u^0(t)| = 1$, we see from the equation that $|\frac{1}{2} - x(t-1)| = 1$ or $x(t-1) = \frac{1}{2}$, $t \in [2,3]$. This implies $x(t) = \frac{1}{2}$ and $\dot{x}(t) = 0$ for $t \in [1,2]$. But then $0 = x(t-1) + u^0(t)$, $t \in [1,2]$ and $|u^0(t)| = 1$ implies $|x(t-1)| = 1$, $t \in [1,2]$ or $|x(t)| = 1$, $t \in [0,1]$, contradicting the fact that $x(1) = \frac{1}{2}$.

Example 4.2. Consider the neutral system

$$\dot{x}(t) = \dot{x}(t-1) + u(t) \quad t \in [0,2]$$

with U as in the previous example and $\xi(t) = 2-t$, $t \in [1,2]$. The assumption that u is bang-bang and ξ is attained leads easily to the conclusion that $|\dot{\phi}(t)| = 1$ or $\dot{\phi}(t) = -3$ where ϕ is the initial function. But it is easy to demonstrate that there are initial functions ϕ (e.g.

$\varphi(t) \equiv 1$) with $|\dot{\varphi}(t)| \neq 1$, $\dot{\varphi}(t) \neq -3$, and admissible controls u such that ζ is attainable from φ using u . Thus there are initial functions φ for which the attainable set in $\mathcal{C}^n[-h, 0]$ from φ using bang-bang controls is a proper subset of the set attained using all admissible controls.

The density theorem mentioned above can also be obtained under less restrictive assumptions as an easy application of a result due to Fattorini. For example, consider (3.3) with $\mu \equiv 0$, $\Phi = \{\varphi\}$, $g(t) = B(t)u(t)$, $B \in L_1$, and define $U = \prod_{i=1}^m [-1, 1]$, $U_e = \prod_{i=1}^m \{-1, 1\}$ so that U_e consists of the vertices of the "cube" U . Let $\mathcal{U} \equiv \{u: [t_0, t_1] \rightarrow \mathbb{R}^m \mid u \text{ measurable, } u(t) \in U\}$ while $\mathcal{U}^\# \equiv \{u: [t_0, t_1] \rightarrow \mathbb{R}^m \mid u \text{ piecewise continuous, } u(t) \in U_e\}$. Then $\mathcal{A}_t(\Phi, \mathcal{U})$, $\mathcal{A}_t(\Phi, \mathcal{U}^\#)$ are defined as subsets of $\mathcal{C}^n[-h, 0]$ in the usual way using the representation results of Section 3. It is not hard to argue that one could make use of Bochner integrals [33] in place of the Lebesgue integrals in defining $\mathcal{A}_t(\Phi, \mathcal{U})$, $\mathcal{A}_t(\Phi, \mathcal{U}^\#)$. Under the standing hypotheses of Section 3, it is then easy to verify that Lemma 1 of [18] is applicable and thus $\mathcal{A}_t(\Phi, \mathcal{U}^\#)$ is dense in $\mathcal{A}_t(\Phi, \mathcal{U})$ in the norm of $\mathcal{C}^n[-h, 0]$.

Remark 4.3. The examples above show that while $\mathcal{A}_t(\Phi, \mathcal{U}^\#)$ may be dense in $\mathcal{A}_t(\Phi, \mathcal{U})$, it will in general be a proper subset of $\mathcal{A}_t(\Phi, \mathcal{U})$ and hence is not closed in $\mathcal{C}^n[-h, 0]$. Referring to Remark 4.1, we see that one therefore would not obtain the closure results of Lemma 4.3 if the convexity assumptions of (4.1) are relaxed. Note that this differs from the situation

for linear ODE and FDE systems where one can obtain existence results in the absence of convexity assumptions when the "state" is taken in \mathbb{R}^n [4, 35, 45].

Section 5. Necessary and Sufficient Conditions.

In this section we shall first derive necessary conditions for the problem given in Section 2 with t_0, t_1 fixed and $\mathcal{T}_0 = \{\varphi\}$, $\mathcal{T}_1 = \{\xi\}$ where φ, ξ are given in $\mathcal{C}^n[-h, 0]$, i.e., the fixed endpoint problem in control theory. Necessary conditions for problems with t_1 variable (including the time optimal problem) and with more general manifolds $\mathcal{T}_0, \mathcal{T}_1$ have been given in [39, 40]. Considering the problem as formulated in Section 2, we define $\hat{f} = (f^0, f)$ and in addition to the standing hypotheses of Section 3 on μ we assume: $\hat{f}: \mathcal{C}^n[t_0-h, t_1] \times U \times [t_0, t_1] \rightarrow \mathbb{R}^{n+1}$ is continuously differentiable in x for each fixed $(u, t) \in U \times [t_0, t_1]$ and Borel measurable in (u, t) for each fixed $x \in \mathcal{C}^n[t_0-h, t_1]$. Furthermore, given any $u \in \mathcal{U}$ and compact convex $X \subset \mathbb{R}^n$, there is an $m \in L_1[t_0, t_1]$ such that $|\hat{f}(x(\cdot), u(t), t)| \leq m(t)$, $|d\hat{f}[x(\cdot), u(t), t; \cdot]| \leq m(t)$ for each $x \in C([t_0-h, t_1], X)$ and $t \in [t_0, t_1]$, where $d\hat{f}$ is the Frechet differential of \hat{f} with respect to x (see [2, p. 1-2]). Defining $x^0(t) \equiv \int_{t_0}^t f^0(x(s), u(s), s) ds$ and $\hat{x} = (x^0, x)$, we have the following necessary conditions that must be satisfied by solutions to the above problem.

Theorem 5.1. Let (x^*, u^*) be optimal. Then there exist $\alpha^0 \leq 0$, $\hat{\psi}: [t_0, \infty) \rightarrow \mathbb{R}^{n+1}$, $\lambda: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ with $\hat{\psi}, \lambda$ of bounded variation and left-continuous such that

- (i) $\lambda = (\lambda_{n+1} - \lambda_1, \lambda_{n+2} - \lambda_2, \dots, \lambda_{2n} - \lambda_n)$ where the λ_j , $j = 1, \dots, 2n$ satisfy: λ_j is constant on $(-\infty, -h]$, $\lambda_j(s) = 0$ for $s > 0$,

λ_j is left-continuous and non-increasing with

$$|\alpha^0| + \sum_{j=1}^{2n} \text{Var}([-h, 0]; \lambda_j) > 0;$$

(ii) $\hat{\psi} = (\psi^0, \psi)$ satisfies $\psi^0 = \alpha^0 \leq 0$ and

$$\begin{aligned} \psi(s) &= \int_{s-t_1}^{0^+} d\lambda(\theta) + \int_s^{t_1^+} d\psi(\theta) \mu(\theta, s) - \int_s^{t_1} [\psi(\theta) \eta^*(\theta, s) - \alpha^0 f_x^{0*}(\theta)] d\theta \\ &\quad s \in [t_0, t_1] \\ \psi(s) &= 0 \quad s > t_1, \end{aligned}$$

where $f_x^{0*}(\theta) = f_x^0(x^*(\theta), u^*(\theta), \theta)$ and η^* is such that

$$df[x^*(\cdot), u^*(t), t; y] = \int_{t_0-h}^t d_s \eta^*(t, s) y(s) \quad \text{for } t \in [t_0, t_1],$$

$y \in \mathcal{C}^n[t_0-h, t_1];$

$$(iii) \int_{t_0}^{t_1} \hat{\psi}(s) \hat{f}(x^*(\cdot), u(s), s) ds \leq \int_{t_0}^{t_1} \hat{\psi}(s) \hat{f}(x^*(\cdot), u^*(s), s) ds$$

for all $u \in \mathcal{U}$.

Proof: Let $\mathcal{E} = \mathcal{E}' \equiv \{\hat{x} \in \mathcal{C}^{n+1}[t_0-h, t_1] \mid \dot{x}^0(t) = f^0(x(t), u(t), t),$
 $\frac{d}{dt}D(x(\cdot), t) = f(x(\cdot), u(t), t), t \in [t_0, t_1], \text{ for some } u \in \mathcal{U}, \text{ and } \hat{x}_{t_0} =$
 $(0, \varphi)\}, \mathcal{G}_0 \equiv \mathcal{C}^1[-h, 0], \text{ and } Z_0 = \{y \in \mathcal{G}_0 \mid y(\theta) < 0, \theta \in [-h, 0]\}.$ De-
 fine mappings $\varphi_0: \mathcal{E} \rightarrow \mathbb{R}^1, \varphi_{-i}: \mathcal{E} \rightarrow \mathcal{G}_0, i = -1, -2, \dots, -2n,$ by $\varphi_0(\hat{x}) =$
 $x^0(t_1), \varphi_{-i}(\hat{x})(\theta) = \zeta^i(\theta) - x^i(t_1 + \theta), \varphi_{-i-n}(\hat{x})(\theta) = x^i(t_1 + \theta) - \zeta^i(\theta), \theta \in$
 $[-h, 0], i = 1, 2, \dots, n,$ where $\zeta = (\zeta^1, \dots, \zeta^n).$ The problem formulated

above is then equivalent to the problem of finding $\hat{z} \in \mathcal{E}$ such that $\varphi_i(\hat{z}) \in \bar{Z}_0$, $i = -1, -2, \dots, -2n$, $\varphi_0(\hat{z}) \leq \varphi_0(\hat{x})$ for all $\hat{x} \in \mathcal{E}$ with $\varphi_i(\hat{x}) \in \bar{Z}_0$, $i = -1, \dots, -2n$. Letting $\mathcal{Z} \equiv R^1 \times \prod_{i=1}^{2n} \mathcal{Z}_0$, $Z \equiv \{\alpha \in R^1 \mid \alpha < 0\} \times \prod_{i=1}^{2n} Z_0$, and $\phi = (\varphi_0 - \varphi_0(\hat{z}), \varphi_{-1}, \dots, \varphi_{-2n})$, one has that a solution to this problem is a (ϕ, Z) extremal [47, Definition 2.2]. Defining $\mathcal{E}_1 \equiv \mathcal{E}$ and $\mathcal{Y} \equiv \mathcal{E}^{n+1}[t_0 - h, t_1]$, it is obvious that Condition 6.1 of [47, p. 75] obtains. Next let $\hat{f}^*(t) = \hat{f}(x^*(\cdot), u^*(t), t)$, $\hat{\mathcal{F}} \equiv \{\hat{g}: [t_0, t_1] \rightarrow R^{n+1} \mid \hat{g}(t) = \hat{f}(x^*(\cdot), u(t), t) \text{ for some } u \in \mathcal{U}\}$, and \hat{Y} be the solution to

$$\begin{aligned} \hat{Y}(s, t) &= E_{n+1} + \int_s^{t^+} d_\alpha \hat{Y}(\alpha, t) \hat{\mu}(\alpha, s) - \int_s^{t^+} \hat{Y}(\alpha, t) \hat{\eta}(\alpha, s) d\alpha \quad s < t, \\ (5.1) \quad \hat{Y}(t, t) &= E_{n+1}, \quad \hat{Y}(s, t) = 0 \quad s > t, \end{aligned}$$

where E_{n+1} is the identity in $\mathcal{L}_{(n+1) \times (n+1)}$,

$$\hat{\mu}(\alpha, s) = \begin{pmatrix} 0 & 0 \\ 0 & \mu(\alpha, s) \end{pmatrix}, \quad \hat{\eta}(\alpha, s) = \begin{pmatrix} 0 & -f_x^{0*}(\alpha) \\ 0 & \eta^*(\alpha, s) \end{pmatrix}.$$

Define the convex set \mathcal{M} by $\mathcal{M} \equiv \{\hat{\delta x} \in \mathcal{E}^{n+1}[t_0 - h, t_1] \mid \hat{\delta x}(t) = 0, t \in [t_0 - h, t_0], \hat{\delta x}(t) = \int_{t_0}^t \hat{Y}(\alpha, t) \hat{\delta f}(\alpha) d\alpha, t \in [t_0, t_1], \hat{\delta f} \in \text{co}(\hat{\mathcal{F}}) - \hat{f}^*\}$. Using the hypotheses of this paper, a generalized idea of quasiconvexity [1, 2, 21, 46, 47], the chattering lemma of Gamkrelidze [21, Lemma 4.1; 2, Lemma 3.1], and reasoning similar to that of Neustadt [46, Section 3], a long and tedious argument verifies that Condition 6.2' of [47, p. 76]

is satisfied. For retarded systems the arguments are much like those in [46; see also 1, 2]. Some technical difficulties are involved for neutral systems, but Kent [39] has shown that these can be overcome. Finally, Condition 6.4 of [47] can easily be shown to hold with $h = (h_0, h_{-1}, \dots, h_{-2n})$ mapping \mathcal{Y} into \mathcal{Z} given by $h_0(\hat{y}) \equiv y^0(t_1)$, $h_{-i}(\hat{y})(\theta) \equiv -y^i(t_1 + \theta)$, $h_{-i-n}(\hat{y})(\theta) \equiv y^i(t_1 + \theta)$, $\theta \in [-h, 0]$, $i = 1, 2, \dots, n$. Thus, Theorem 6.2 and hence Theorem 3.1 of [47] hold. We shall show that the desired results follow from Theorem 3.1.

Applying Theorem 3.1 of [47] we obtain the existence of a non-zero $l \in \mathcal{Z}^*$ such that

$$(5.2) \quad l(\hat{x}) \geq 0 \quad \text{for all } \hat{x} \in \bar{\mathcal{Z}}$$

$$(5.3) \quad l \circ h(\hat{\delta x}) \leq 0 \quad \text{for all } \hat{\delta x} \in \mathcal{M}.$$

From the definition of \mathcal{Z} , we see that there are $\alpha^0 \in \mathbb{R}^1$ and $l_{-i} \in \mathcal{Z}_0^*$, $i = 1, 2, \dots, 2n$ such that

$$l(\omega, \xi_1, \dots, \xi_{2n}) = \alpha^0 \omega + \sum_{i=1}^{2n} l_{-i}(\xi_i)$$

for every $(\omega, \xi_1, \dots, \xi_{2n}) \in \mathcal{Z}$. Since $\mathcal{Z}_0 = \mathcal{C}^1[-h, 0]$, it follows that there are $\lambda_i: (-\infty, \infty) \rightarrow \mathbb{R}^1$ of bounded variation, with λ_i left continuous,

$\lambda_i(s) = 0$ for $s > 0$, λ_i constant on $(-\infty, -h]$, $i = 1, 2, \dots, 2n$ such that

$$l_{-i}(\xi) = \int_{-h}^{0^+} d\lambda_i(\theta) \xi(\theta)$$

for every $\xi \in \mathcal{L}^1[-h, 0]$. Observing that $(-1, 0, \dots, 0), (0, \xi, 0, \dots, 0), \dots, (0, 0, \dots, \xi)$ are in \bar{Z} whenever $\xi \in \bar{Z}_0$ we conclude from (5.2) that $\alpha^0 \leq 0$ and λ_i is non-increasing, $i = 1, 2, \dots, 2n$. The fact that l is non-zero yields $|\alpha^0| + \sum_{i=1}^{2n} \text{Var}([-h, 0]; \lambda_i) > 0$.

The inequality (5.3) thus becomes

$$\begin{aligned} \alpha^0 \delta x^0(t_1) + \sum_{i=1}^n \int_{-h}^{0^+} d\lambda_i(\theta) [-\delta x^i(t_1 + \theta)] \\ + \sum_{i=1}^n \int_{-h}^{0^+} d\lambda_{n+i}(\theta) [\delta x^i(t_1 + \theta)] \leq 0 \end{aligned}$$

for all $\hat{\delta x} \in \mathcal{M}$. Defining $\lambda: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ as in statement (i) above, this then becomes

$$(5.4) \quad \alpha^0 \delta x^0(t_1) + \int_{-h}^{0^+} d\lambda(\theta) \delta x(t_1 + \theta) \leq 0$$

for all $\hat{\delta x} \in \mathcal{M}$. The matrix solution \hat{Y} in (5.1) may be written $\hat{Y} = \begin{pmatrix} 1 & Y^0 \\ 0 & Y \end{pmatrix}$

where $Y^0(s, t)$ is an n -dimensional row vector and $Y(s, t)$ is in $\mathcal{L}_{n \times n}$.

The equations for Y^0, Y are

$$Y^0(s, t) = \int_s^{t^+} d_\alpha Y^0(\alpha, t) \mu(\alpha, s) - \int_s^t [-f_x^{0*}(\alpha) + Y^0(\alpha, t) \eta^*(\alpha, s)] d\alpha$$

and

$$Y(s, t) = E_n + \int_s^{t^+} d_\alpha Y(\alpha, t) \mu(\alpha, s) - \int_s^t Y(\alpha, t) \eta^*(\alpha, s) d\alpha$$

for $s \leq t$ with $Y^0(s, t)$, $Y(s, t)$ vanishing for $s > t$. Using the definition of \mathcal{M} in (5.4) we obtain

$$\alpha^0 \int_{t_0}^{t_1} [\delta f^0(s) + Y^0(s, t_1) \delta f(s)] ds + \int_{-h}^{0^+} d\lambda(\theta) \left[\int_{t_0}^{t_1 + \theta} Y(s, t_1 + \theta) \delta f(s) ds \right] \leq 0$$

for all $\hat{\delta f}$ in $\text{co}(\hat{\mathcal{F}}) - \hat{f}^*$. This can be written as

$$\int_{t_0}^{t_1} [\alpha^0 \delta f^0(s) + \{\alpha^0 Y^0(s, t_1) + \int_{-h}^{0^+} d\lambda(\theta) Y(s, t_1 + \theta)\} \delta f(s)] ds \leq 0.$$

Defining $\psi(s) \equiv \alpha^0 Y^0(s, t_1) + \int_{-h}^{0^+} d\lambda(\theta) Y(s, t_1 + \theta)$ and $\psi^0 = \alpha^0$, an appropriate choice of $\hat{\delta f}$ yields (iii). The properties of \hat{Y} guaranteed by Theorem 3.1 of Section 3 of this paper lead easily to the conclusion that ψ is left-continuous and of bounded variation on $[t_0, \infty)$ with $\psi(s) = 0$ for $s > t_1$. Using the equations for Y^0, Y in the definition of ψ and making several interchanges in the orders of integration (it is here that the Borel measurability of \hat{Y} becomes important - see Remark 3.1 above) which are justified by the Fubini type theorem in [10], we easily conclude that ψ satisfies the equation in (ii). This completes the proof of Theorem 5.1.

We point out that the equation for ψ in (ii) can be written in the equivalent form

$$\psi(s) = -\lambda(s-t_1) + \int_s^{t_1^+} d\psi(\theta)\mu(\alpha, s) - \int_s^{t_1} [\psi(\theta)\eta^*(\theta, s) - \alpha^{O_f O_x^*}(\theta)]d\theta.$$

Since λ is usually only of bounded variation (BV) on $[-h, 0]$, one would scarcely expect ψ to be smoother, say absolutely continuous (AC), on $[t_1-h, t_1]$. But for $s < t_1-h$, the $\lambda(s-t_1)$ term is a constant ($\lambda(-h)$) and one might ask whether the equation can be written in differentiated form (i.e., is ψ AC?) for $s \in [t_0, t_1-h]$. Even for simple neutral systems ψ is not in general AC. For example if the system in the problem has the form (2.3) the equation for ψ becomes

$$\begin{aligned} \psi(s) = & -\lambda(s-t_1) - \int_{s+h}^{t_1^+} d\psi(\theta)A(\theta) + \int_{s+h}^{t_1} \psi(\theta)C(\theta)d\theta \\ & + \int_s^{t_1} [\psi(\theta)B(\theta) + \alpha^{O_f O_x^*}(\theta)]d\theta \end{aligned}$$

and it is easily seen that ψ has jump discontinuities at $s = t_1-h, t_1-2h, t_1-3h, \dots$. However for problems involving system (2.4) the ψ equation is

$$\psi(s) = -\lambda(s-t_1) + \int_{s+h}^{t_1} \psi(\theta)C(\theta)d\theta + \int_s^{t_1} [\psi(\theta)B(\theta) + \alpha^{O_f O_x^*}(\theta)]d\theta$$

and ψ is readily seen to be AC on $[t_0, t_1-h]$ satisfying a.e.

$$\dot{\psi}(s) = -\alpha^0 f_x^{0*}(s) - \psi(s)B(s) - \psi(s+h)C(s+h).$$

In fact there are a number of types of retarded systems for which the corresponding ψ will be AC on $[t_0, t_1-h]$. These are discussed on p. 15-16 of [2]. Note that for retarded systems the associated ψ equation is the same as the $\bar{\lambda}$ equation of (i) of Theorem 1 in [2, p. 3] with the exception of the $\lambda(s-t_1)$ term which will be constant for $s < t_1-h$.

If, instead of $\zeta \in \mathcal{L}^n[-h, 0]$, one only specifies q components ($q < n$) of x on $[t_1-h, t_1]$, then one can derive the corresponding necessary conditions as above by using $2q$ of the φ_{-i} constraint functions instead of $2n$. The ideas for use of the "conflict" constraints φ_{-i} evolved from trying to treat the terminal conditions in $\mathcal{L}^n[-h, 0]$ as some type of bounded state variable restraint. It is not surprising then that in a number of examples [39], the multiplier λ behaves much like the multipliers obtained in considering bounded state variable problems [8, 48]. That is, λ has jumps at t_1-h, t_1 (the points at which the trajectory enters and leaves the "boundary" ζ of the restraint region), and has jumps at points where the trajectory follows the "boundary" across a point where it is not smooth. The multiplier λ can also be interpreted as a terminal boundary condition in function space [36, 38] which usually results from transversality conditions.

Let us mention briefly other ways of deriving the necessary conditions of Theorem 5.1. First, various other types of "conflicting" con-

straints [39] will yield essentially the same theorem; also it is possible to use a constraint of the form $\varphi_{-1}(\hat{x}) \equiv |x_{t_1} - \xi| - \varepsilon$ (an ε -ball about ξ in $\mathcal{C}^n[-h, 0]$) together with Theorem 5.1 of [47]. Allowing $\varepsilon \rightarrow 0$ and using the linear "subfunctionals" Λ of that theorem yield (formally) the same necessary conditions as derived above. All of these proofs yield necessary conditions that have a notable deficiency. Observe that for the maximum principle (iii) to be nontrivial, one needs α^0 or λ or both non-zero. The non-zero statement involving ℓ in the proof (see also (i) of the theorem) does not guarantee this. In fact, nothing rules out the situation $\alpha^0 = 0$ and $\lambda_i = \lambda_{n+i}$, $i = 1, 2, \dots, n$, in which case the inequality in (iii) is trivially true with $\psi \equiv 0$, $\alpha^0 = 0$. This difficulty also appears in the approach employing Theorem 5.1 of [47] mentioned above (there is no guarantee that $\Lambda \neq 0$). Nonetheless, as we shall see, the conditions of Theorem 5.1 above are non-trivial in numerous examples [39] and are actually sufficient for linear problems with certain payoffs when $\alpha^0 \neq 0$ (normality). Thus we do obtain conditions which are both necessary and sufficient for a non-empty class of normal problems.

There are derivations of the necessary conditions presented here which for special problems do yield $\alpha^0 \neq 0$ (and hence nontriviality and sufficiency). The authors have shown that using an attainable sets approach [42] in function space for linear retarded systems with integral quadratic payoff and $\mathcal{U} = L_2$, necessary conditions which are the same as those of

Theorem 5.1 can be obtained. Jacobs and Kao [36, 38] have obtained equivalent necessary conditions for problems with general nonlinear retarded systems by employing an abstract Lagrange Multiplier Rule [43, p. 243] in the function space $W_2^{(1)}$. But both of these latter approaches are for unconstrained controls and require more restrictive hypotheses than are desirable (roughly speaking, the matrix $D(t)$, where $k(u(t), t) = D(t)u(t)$, has rank n for a.e. t).

We next exhibit a class of problems for which the necessary conditions derived above are also sufficient. Let $f^0(z, u, t) = g^0(z, t) + k^0(u, t)$ and $f(x(\cdot), u, t) = \int_{t_0-h}^t d_s \eta(t, s)x(s) + k(u, t)$ satisfy the hypotheses given preceding Theorem 5.1 where η is as in (3.4). In addition assume that $g^0: R^n \times [t_0, t_1] \rightarrow R^1$ is convex in z for each fixed $t \in [t_0, t_1]$. Under these hypotheses we have the following sufficiency results.

Theorem 5.2. Suppose (x^*, u^*) satisfy the conditions of Theorem 5.1 with $\psi^0 = \alpha^0 < 0$. Then (x^*, u^*) are optimal.

Proof: Let $v \in \mathcal{U}$ be such that the corresponding trajectory $x = x(v)$ satisfies $x_{t_1} = \zeta_1 = x_{t_1}^*$. Then from the equations for x, x^* we have

$$\begin{aligned} 0 = & \int_{t_0}^{t_1^+} d\psi(s) \left\{ [x(s) - x^*(s)] - \int_{t_0}^{t_1^+} d_\theta \mu(s, \theta) [x(\theta) - x^*(\theta)] \right. \\ & \left. - \int_{t_0}^s \int_{t_0}^{t_1^+} d_\theta \eta(\sigma, \theta) [x(\theta) - x^*(\theta)] d\sigma - \int_{t_0}^s [k(v(\sigma), \sigma) - k(u^*(\sigma), \sigma)] d\sigma \right\}. \end{aligned}$$

Thus, using the definition of x^0 we obtain

$$\begin{aligned}
-\alpha^0[x^{0*}(t_1)-x^0(t_1)] &= \alpha^0 \int_{t_0}^{t_1} [g^0(x(s),s)-g^0(x^*(s),s)]ds \\
&\quad + \alpha^0 \int_{t_0}^{t_1} [k^0(v(s),s) - k^0(u^*(s),s)]ds \\
&\quad + \int_{t_0}^{t_1^+} d\psi(s)[x(s)-x^*(s)] - \int_{t_0}^{t_1^+} d\psi(s) \int_{t_0}^{t_1^+} d_\theta \mu(s,\theta)[x(\theta)-x^*(\theta)] \\
&\quad - \int_{t_0}^{t_1^+} d\psi(s) \left\{ \int_{t_0}^s \int_{t_0}^{t_1^+} d_\theta \eta(\sigma,\theta)[x(\theta)-x^*(\theta)]d\sigma \right\} \\
&\quad - \int_{t_0}^{t_1^+} d\psi(s) \int_{t_0}^s [k(v(\sigma),\sigma)-k(u^*(\sigma),\sigma)]d\sigma.
\end{aligned}$$

Adding and subtracting a term involving $g_x^{0*}(s) \equiv g_x^0(x^*(s),s)$, integrating by parts in the last two terms, and noting that $\psi(t_1^+) = 0$, we find

$$\begin{aligned}
-\alpha^0[x^{0*}(t_1)-x^0(t_1)] &= \alpha^0 \int_{t_0}^{t_1} \{g^0(x(s),s)-g^0(x^*(s),s)-g_x^{0*}(s)[x(s)-x^*(s)]\}ds \\
&\quad + \int_{t_0}^{t_1} \left\{ \alpha^0 [k^0(v(s),s)-k^0(u^*(s),s)] + \psi(s)[k(v(s),s)-k(u^*(s),s)] \right\} ds \\
&\quad + \int_{t_0}^{t_1^+} d\psi(s)[x(s)-x^*(s)] - \int_{t_0}^{t_1^+} d\psi(s) \int_{t_0}^{t_1^+} d_\theta \mu(s,\theta)[x(\theta)-x^*(\theta)] \\
&\quad + \int_{t_0}^{t_1} d\psi(s) \int_{t_0}^{t_1^+} d_\theta \eta(s,\theta)[x(\theta)-x^*(\theta)]ds \\
&\quad + \alpha^0 \int_{t_0}^{t_1} g_x^{0*}(s)[x(s)-x^*(s)]ds.
\end{aligned}$$

Using the Fubini type theorem [10] to interchange the order of integration in several of the integrals and combining terms, we have

$$\begin{aligned} -\alpha^0[x^{0*}(t_1)-x^0(t_1)] &= \alpha^0 \int_{t_0}^{t_1} \{g^0(x(s),s)-g^0(x^*(s),s)-g_x^{0*}(s)[x(s)-x^*(s)]\}ds \\ &\quad + \int_{t_0}^{t_1} \hat{\psi}(s)[\hat{f}(x^*(\cdot),v(s),s) - \hat{f}(x^*(\cdot),u^*(s),s)]ds \\ &\quad + \int_{t_0}^{t_1} d_\theta \left\{ \psi(\theta) - \int_{t_0}^{t_1} \psi(s)\mu(s,\theta) + \int_{t_0}^{t_1} \psi(s)\eta(s,\theta)ds - \int_{\theta}^{t_1} \alpha^0 g_x^{0*}(s)ds \right\} [x(\theta)-x^*(\theta)]. \end{aligned}$$

Noting that $\int_{t_0}^{t_1} d_\theta \psi(s)\mu(s,\theta) = \int_{\theta}^{t_1} d_\theta \psi(s)\mu(s,\theta)$ and $\int_{t_0}^{t_1} \psi(s)\eta(s,\theta)ds =$

$\int_{\theta}^{t_1} \psi(s)\eta(s,\alpha)ds$, it follows from the convexity assumption on g^0 and parts

(ii) and (iii) of Theorem 5.1 that

$$\begin{aligned} -\alpha^0[x^{0*}(t_1)-x^0(t_1)] &\leq \int_{t_0}^{t_1} d_\theta \lambda(\theta-t_1)[x(\theta)-x^*(\theta)] \\ &= \int_{t_1-h}^{t_1} d_\theta \lambda(\theta-t_1)[x(\theta)-x^*(\theta)] \\ &= 0, \end{aligned}$$

or since $\alpha^0 < 0$, $[x^{0*}(t_1)-x^0(t_1)] \leq 0$. Thus (x^*, u^*) is optimal.

The underlying idea for this sufficiency proof can be traced at least as far back as a paper by Rozonoer [50]. A number of other authors [20, 22, 41, 42] have used and refined it. In fact, in a recent work of Funk and Gilbert [20] it is pointed out that under normality and certain convexity assumptions, the abstract necessary conditions derived by Neustadt [46, 47] are also sufficient in a number of situations.

Section 6. Examples.

We first present two examples which illustrate use of some of the results obtained above.

Example 6.1. Consider a system described by the scalar equation

$$\dot{x}(t) = x(t) - x(t-1) + u(t).$$

We wish to drive from the initial function $\varphi = 0$ to the target function given by $\xi(\theta) = 2+\theta$, $\theta \in [-1,0]$, while minimizing $J(u) = \int_0^3 x(t)dt$ subject to $U = [-1,1]$.

Since $f^0 = x$, the maximum condition of Theorem 5.1 reduces to

$$\int_0^3 \psi(s)u(s)ds \leq \int_0^3 \psi(s)u^*(s)ds.$$

Thus $u^*(s) = \text{sgn}[\psi(s)]$ whenever $\psi(s) \neq 0$. The equation for ψ reduces to

$$\psi(s) = -\lambda(s-3) + \int_{s+1}^3 \psi(\theta)(-1)d\theta + \int_s^3 [\psi(\theta)(1)+\alpha^0]d\theta, \quad s \in [0,3].$$

From the discussion and examples of Section 4, we guess that u^* is not bang-bang on $[2,3]$. Thus we try $\psi \equiv 0$ on $(2,3]$ and $\alpha^0 = -1$. On $(2,3]$ the equation for ψ becomes

$$0 = -\lambda(s-3) + \int_s^3 (-1)d\theta, \quad \text{so}$$

$$\lambda(\theta) = \theta \quad \text{for } \theta \in (-1, 0].$$

On $[1, 2]$ we differentiate the equation for ψ and obtain

$$\dot{\psi}(s) = -\psi(s) - \alpha^0 = -\psi(s) + 1.$$

Thus $\psi(s) = ce^{-(s-2)} + 1$ on $[1, 2]$. But $\psi(2) = -\lambda(-1) + \int_2^3 (-1)d\theta = -\lambda(-1) - 1$, so $c+1 = -1-\lambda(-1)$. Let $v = -\lambda(-1)$; then $c = v-2$, and

$$\psi(s) = (v-2)e^{(2-s)} + 1, \quad s \in [1, 2].$$

Similarly, on $[0, 1]$ $\dot{\psi}(\theta) = -\psi(\theta) + \psi(\theta+1) + 1$. Integrating from $\psi(1) = (v-2)e+1$, we obtain

$$\psi(\theta) = (v-2)\{e^{(2-\theta)} - (1-\theta)e^{(1-\theta)}\} + 2 - e^{(1-\theta)}, \quad \theta \in [0, 1].$$

We claim $\psi(0) < 0$. Suppose $\psi(0) \geq 0$. Then $(v-2)[e^2 - e] + 2 - e \geq 0$, or

$$(v-2) \geq \frac{e-2}{e^2-e} > 0. \quad \text{Thus } \psi(s) = (v-2)e^{(2-s)} + 1 > 1 \quad \text{for } s \in [1, 2],$$

$\dot{\psi}(\theta) > -\psi(\theta) + 1 + 1 = -\psi(\theta) + 2$ for $\theta \in [0, 1]$. Hence if $\psi(0) \geq 0$, then

$\psi(s) > 0$ on $(0, 2)$, and $u^*(s) = +1$ for $s \in (0, 2)$. A simple integration shows that this implies $x^*(2) > 1$, contradicting the boundary condition

$x(2) = 1$ and proving the claim. We next claim that $\psi(1) > 0$. Suppose $\psi(1) \leq 0$. Then $(v-2)e+1 \leq 0$, or $v-2 \leq -\frac{1}{e}$. For $\theta \in (0,1)$,

$$\begin{aligned}\psi(\theta) &= (v-2)[e^{(1-\theta)\{e+\theta-1\}}] + 2 - e^{(1-\theta)} \\ &\leq -e^{-1}[e^{(1-\theta)\{e+\theta-1\}}] + 2 - e^{(1-\theta)} \\ &= -e^{-\theta\{e+\theta-1+e\}} + 2 = -e^{-\theta\{2e+\theta-1\}} + 2.\end{aligned}$$

This expression has its maximum over $[0,1]$ at $\theta = 1$,

$$\psi(1) \leq -e^{-1}\{2e^1\} + 2 = 0,$$

and hence $\psi(\theta) < 0$ for $\theta \in (0,1)$. For $s \in (1,2)$,

$$\begin{aligned}\dot{\psi}(s) &= -\psi(s) + 1 = -(v-2)e^{(2-s)} \\ &\geq e^{-1}e^{(2-s)} = e^{(1-s)} > 0.\end{aligned}$$

Thus ψ has at most one zero in $[1,2]$. If there is no zero in $[1,2]$, $u^*(s) = -1$ for all $s \in (0,2)$, and a simple integration shows that the response to such a control does not satisfy $x^*(2) = 1$. If ψ has a zero at $\omega \in [1,2]$, then u^* is given on $(0,2)$ by

$$u^*(s) = \begin{cases} -1, & s \in (0, \omega), \\ +1, & s \in (\omega, 2). \end{cases}$$

Another integration shows that the response to such a control also can not satisfy $x^*(2) = 1$. Thus $\psi(1) > 0$ must hold.

From $\psi(1) > 0$ it follows that $(v-2)e^{1+1} > 0$, or $(v-2)e^1 > -1$.

On $(1, 2)$

$$\begin{aligned} \psi(s) &= (v-2)e^{(2-s)+1} = (v-2)e^1 e^{(1-s)+1} \\ &> -e^{(1-s)+1}, \end{aligned}$$

so $\psi(s) > 0$ for $s \in (1, 2)$. On $(0, 1)$,

$$\dot{\psi}(\theta) = -\psi(\theta) + \psi(\theta+1)+1 > -\psi(\theta)+1.$$

Thus we have that ψ has at most one zero in $(0, 1)$. Since ψ is continuous with $\psi(0) < 0$ and $\psi(1) > 0$, ψ has a zero in $(0, 1)$, which we denote by ω . This implies the optimal control is given by

$$u^*(s) = \begin{cases} -1, & s \in (0, \omega), \\ +1, & s \in (\omega, 2). \end{cases}$$

The response to this control is

$$x^*(t) = \begin{cases} 1-e^t & , t \in [0, \omega] \\ 2e^{(t-\omega)} - e^t - 1 & , t \in [\omega, 1] \\ 2e^{(t-\omega)} - e^t + (t-2)e^{(t-1)} & , t \in [1, 1+\omega] \\ 2e^{(t-\omega)} - e^t + (t-2)e^{(t-1)} - 2(t-2-\omega)e^{(t-1-\omega)} - 2, & t \in [1+\omega, 2]. \end{cases}$$

From the boundary conditions

$$1-x^*(2) = 2e^{(2-\omega)} - e^2 + 2\omega e^{(1-\omega)} - 2$$

$$2(e+\omega)e^{(1-\omega)} = 3+e^2$$

$$(e+\omega)e^{(1-\omega)} = \frac{1}{2}(3+e^2).$$

The solution is approximately $\omega = 0.531$. To determine v we go back to the equation $\psi(\omega) = 0$.

$$(v-2)\{e^{(2-\omega)} - (1-\omega)e^{(1-\omega)}\} + 2 - e^{(1-\omega)} = 0$$

$$v = 2 - \frac{2 - e^{(1-\omega)}}{e^{(2-\omega)} - (1-\omega)e^{(1-\omega)}} \approx 2 - 0.111 = 1.889.$$

We determine u^* on $(2, 3)$ by using the equation for x and the fact that $x(s) = s-1$, $s \in (2, 3)$. On $(2, 3)$, $1 = s-1 - x(s-1) + u^*(s)$, $u^*(s) = 2-s + x(s-1)$, from which it follows that

$$u^*(s) = \begin{cases} 2-s+2e^{(s-1-\omega)} - e^{(s-1)} + (s-3)e^{(s-2)} & , s \in (2, 2+\omega) \\ -s+2e^{(s-1-\omega)} - e^{(s-1)} + (s-3)e^{(s-2)} - 2(s-3-\omega)e^{(s-2-\omega)} & , s \in (2+\omega, 3). \end{cases}$$

Since $|u^*| \leq 1$ this is an admissible control satisfying the necessary conditions with $\alpha^0 \neq 0$, and by Theorem 5.2 u^* is an optimal control. It is not difficult to argue that u^* is in fact the unique optimal control.

Example 6.2. Consider a system described by the scalar equation

$$\dot{x}(t) = \dot{x}(t-1) + u(t).$$

We wish to drive from the initial function $\varphi = 0$ to the target function given by

$$\zeta(\theta) = \begin{cases} -\frac{1}{2} - \theta, & \theta \in [-1, -\frac{1}{2}], \\ \frac{1}{2} + \theta, & \theta \in [-\frac{1}{2}, 0], \end{cases}$$

while minimizing $J(u) = \int_0^3 u^2(t) dt$ subject to $U = R^1$.

The maximum condition of theorem 5.1 reduces to

$$\alpha^0 [v^2 - u^{*2}(s)] + \psi(s) [v - u^*(s)] \leq 0$$

for all $v \in R^1$, almost all $s \in [0, 3]$. Let us choose $\alpha^0 = -1$. Then this implies

$$u^*(s) = \frac{1}{2} \psi(s).$$

The equation for ψ reduces to

$$\psi(s) = \psi(s+1) - \lambda(s-3), \quad s \in [0,3],$$

hence we have that

$$u^*(s) = u^*(s+1) - \frac{\lambda(-1)}{2} \text{ for } s \in (0,2).$$

From the shape of the target, the equations, and this relationship between $u^*(s)$ and $u^*(s+1)$, we guess that the optimal trajectory is composed of straight line segments of time-length $\frac{1}{2}$. Let the slopes of these be α , β , γ , δ respectively on $[0,2]$, and set $-\frac{1}{2}\lambda(-1) = \kappa$. Thus

$$u^*(s) = \begin{cases} \alpha, & s \in (0, \frac{1}{2}) \\ \beta, & s \in (\frac{1}{2}, 1) \\ \gamma - \alpha, & s \in (1, \frac{3}{2}) \\ \delta - \beta, & s \in (\frac{3}{2}, 2) \\ -1 - \gamma, & s \in (2, \frac{5}{2}) \\ 1 - \delta, & s \in (\frac{5}{2}, 3). \end{cases}$$

Substituting these in $u^*(s) = u^*(s+1) + \kappa$, we obtain the systems of equations

$$\delta - \beta = 1 - \delta + \kappa, \quad \gamma - \alpha = -1 - \gamma + \kappa$$

$$\beta = \delta - \beta + \kappa, \quad \alpha = \gamma - \alpha + \kappa$$

Solving in pairs; $\beta = 2\delta - 1 - \kappa$ and $\delta = 2\beta - \kappa$ imply $\beta = 4\beta - 2\kappa - 1 - \kappa$, or $3\beta = 3\kappa + 1$. Thus $\beta = \kappa + \frac{1}{3}$, $\delta = \kappa + \frac{2}{3}$. Also $\alpha = 2\gamma - \kappa + 1$ and $\gamma = 2\alpha - \kappa$ imply $\alpha = 4\alpha - 2\kappa + 1 - \kappa$, or $3\alpha = 3\kappa - 1$. Thus $\alpha = \kappa - \frac{1}{3}$, $\gamma = \kappa - \frac{2}{3}$. The endpoint $x^*(2) = \frac{1}{2}$ is half the sum of the slopes, so $\frac{1}{2} = \frac{1}{2}(\alpha + \beta + \gamma + \delta)$, $1 = 4\kappa$, or $\kappa = \frac{1}{4}$. Thus $\alpha = -\frac{1}{12}$, $\beta = \frac{7}{12}$, $\gamma = -\frac{5}{12}$, and $\delta = \frac{11}{12}$. From $u^*(s) = \frac{1}{2}\psi(s)$ and the expression for u^* in terms of α , β , γ , and δ we obtain

$$\psi(s) = \begin{cases} -\frac{1}{6}, & s \in (0, \frac{1}{2}) \\ \frac{7}{6}, & s \in (\frac{1}{2}, 1) \\ -\frac{4}{6}, & s \in (1, \frac{3}{2}) \\ \frac{4}{6}, & s \in (\frac{3}{2}, 2) \\ -\frac{7}{6}, & s \in (2, \frac{5}{2}) \\ \frac{1}{6}, & s \in (\frac{5}{2}, 3). \end{cases}$$

This ψ satisfies the equation in Theorem 5.1 with

$$\lambda(\theta) = \begin{cases} -\frac{1}{4}, & \theta \in (-\infty, -1] \\ \frac{7}{6}, & \theta \in (-1, -\frac{1}{2}] \\ -\frac{1}{6}, & \theta \in (-\frac{1}{2}, 0] \\ 0, & \theta \in (0, \infty). \end{cases}$$

The x^* , u^* obtained here satisfy the necessary conditions with ψ and λ given above and $\alpha^0 = -1$. By Theorem 5.2, u^* is optimal. Since the system equations are linear and f^0 is strictly convex in u , standard arguments [42] show that u^* is unique.

There are several ways in which one could attempt to solve fixed function target problems without using the necessary conditions for driving to a function. The most direct is to minimize $\int_{t_0}^{t_1-h} f^0(x(s), u(s), s) ds$ while driving to $x^*(t_1-h)$, and then attempt to determine u^* on (t_1-h, t_1) so that $x_{t_1}^*$ is as desired. If $f^0(x, u, s)$ is independent of u for $s \in (t_1-h, t_1)$ and $U = \mathbb{R}^m$, this approach can succeed provided that given x on $[t_0-h, t_1]$ one can solve the system equations for $u(s)$, $s \in (t_1-h, t_1)$. If $U \neq \mathbb{R}^m$ the method may fail because the u^* determined on (t_1-h, t_1) in this manner need not lie in U ; in the case of example 6.1 it succeeds. In fact, given the assumption that $\psi(s) = 0$ for $s \in (t_1-h, t_1]$ in Example 6.1 the subsequent arguments are identical to those resulting from applying the maximum principle for point-target problems [2, 40] as suggested above. If $f^0(x, u, s)$ is not independent of u for $s \in (t_1-h, t_1)$ this approach will generally fail. In example 6.2, driving to $x(2) = \frac{1}{2}$ while minimizing $\int_0^2 u^2(t) dt$ yields

$$u^*(s) = \begin{cases} \frac{1}{5}, & s \in (0, 1) \\ \frac{1}{10}, & s \in (1, 2), \end{cases} \quad x^*(t) = \begin{cases} \frac{t}{5}, & t \in [0, 1] \\ \frac{3t-1}{10}, & t \in [1, 2], \end{cases}$$

which is not even close to the optimal trajectory for the function target

problem.

A more complicated method involves making the function-target problem equivalent to a point-target problem by altering the form of f^0 on $[t_0, t_1-h]$ to include the "cost" due to the u^* determined on (t_1-h, t_1) as above. With $U = R^m$ one may then use either the necessary conditions of the calculus of variations (see [16, 34, 51]) or the point-target maximum principle. Examples illustrating this can be found in [39]. In example 6.2, the altered cost-functional is

$$\begin{aligned} J(u) &= \int_0^1 u^2(t) dt + \int_1^{3/2} \{u^2(t) + [-1-u(t)-u(t-1)]^2\} dt \\ &\quad + \int_{3/2}^2 \{u^2(t) + [1-u(t)-u(t-1)]^2\} dt \\ &= \int_0^1 \dot{x}^2(t) dt + \int_1^{3/2} \{[\dot{x}(t)-\dot{x}(t-1)]^2 + [-1-\dot{x}(t)]^2\} dt \\ &\quad + \int_{3/2}^2 \{[\dot{x}(t)-\dot{x}(t-1)]^2 + [1-\dot{x}(t)]^2\} dt. \end{aligned}$$

With this form of $J(u)$ in terms of u , one clearly would prefer not to use the maximum principle on this equivalent problem. Applying the modified Euler conditions to the second expression for $J(u)$, one obtains that the optimal trajectory must consist of line segments of time-length $1/2$. However, the relationships between the four slopes and a single constant are not obtained, and the problem of determining the slopes remains difficult without the function-target necessary conditions. When $U \neq R^m$ the

problems encountered in the first point-target method will also occur here.

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